

Lie algebroids associated to Poisson actions

Jiang-Hua Lu

Department of Mathematics, University of Arizona, Tucson, AZ 85721

February 8, 2008

1 Introduction

This work is motivated by a result of Drinfeld in [Dr2]. Recall [Dr1] that a **Poisson Lie group** is a Lie group G together with a Poisson structure such that the group multiplication map

$$G \times G \longrightarrow G$$

is a Poisson map. Given a Poisson Lie group G and a Poisson manifold P , an action

$$\sigma : G \times P \longrightarrow P$$

of G on P is called a **Poisson action** if the action map σ is a Poisson map. When the action is transitive, we say that P is a **Poisson homogeneous G -space**. Poisson G -spaces are the semi-classical analogs of quantum spaces with quantum group actions. Special cases of Poisson homogeneous G -spaces can be found in [Da-So] [Lu1] [Za].

Let P be a Poisson homogeneous G -space. In [Dr2], Drinfeld shows that corresponding to each $p \in P$, there is a maximal isotropic Lie subalgebra \mathfrak{l}_p of the Lie algebra \mathfrak{d} , the double Lie algebra of the tangent Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ of G . Moreover, for $g \in G$, the two Lie algebras \mathfrak{l}_p and \mathfrak{l}_{gp} are related by $\mathfrak{l}_{gp} = Ad_g \mathfrak{l}_p$ via the Adjoint action of G on \mathfrak{d} . In particular, they are isomorphic as Lie algebras. The Lie algebra \mathfrak{l}_p determines the Poisson structure on P , and it can be used to classify Poisson homogeneous G -spaces [Dr2].

The purpose of this note is to find an invariant setting for these Lie algebra \mathfrak{l}_p 's. We construct, for every Poisson manifold P with a Poisson G -action (not necessarily G -homogeneous), a Lie algebroid structure on the direct sum vector bundle $(P \times \mathfrak{g}) \oplus T^*P$

over P . This Lie algebroid will be denoted by $A = (P \times \mathfrak{g}) \bowtie T^*P$ to indicate the fact that it is built out of the two Lie algebroids $P \times \mathfrak{g}$ and T^*P and a pair of representations of them on each other. Moreover, the Lie algebroid A is naturally equipped with an action of G , making it into a Harish-Chandra Lie algebroid of G [B-B]. It follows immediately that the kernel \mathfrak{l}_p of the anchor map of A at each $p \in P$ has a natural Lie algebra structure, and $\mathfrak{l}_{gp} = g \cdot \mathfrak{l}_p$, where $g \cdot \mathfrak{l}_p$ comes from the action of G on A .

When the anchor map of A has full rank everywhere (it is said to be transitive in such a case), the subbundle of A defined by the kernel of the anchor map is a Lie algebra bundle in the sense that local trivialisations exist [Mc1]. This is the case when P is G -homogeneous or when P is symplectic. In both cases, each Lie algebra \mathfrak{l}_p can be naturally embedded in the Lie algebra \mathfrak{d} , the double of the tangent Lie bialgebra of G , as a maximal isotropic Lie subalgebra.

When P is G -homogeneous, these maximal isotropic Lie subalgebras of \mathfrak{d} are precisely those described in [Dr2].

When P is symplectic, we show that the Lie algebra \mathfrak{l}_p 's define an isomorphic family of Lie bialgebras over \mathfrak{g}^* .

As further applications, we describe the symplectic leaves of a Poisson homogeneous G -space P in terms of the Lie algebra \mathfrak{l}_p . We show that the G -invariant Poisson cohomology of P can be realized as relative Lie algebra cohomology of \mathfrak{l}_p relative to the stabilizer subgroup of G at p . As an example, we calculate the K -invariant Poisson cohomology of the Bruhat Poisson structure [Lu-We] on the generalized flag manifold K/T , where K is a compact semisimple Lie group and $T \subset K$ is a maximal torus of K , and we show that there is exactly one cohomology class for each element in the Weyl group of K . We also describe the Lie groupoid corresponding to the Lie algebroid $A = (P \times \mathfrak{g}) \bowtie T^*P$ and show that it is an example of a double Lie groupoid in the sense of Mackenzie [Mc2].

Acknowledgement The author would like to thank Yvette Kosmann-Schwarzbach for drawing her attention to [Dr2] and Sam Evens for answering many questions. She also would like to thank Kirill Mackenzie for helpful comments.

2 Lie algebroids

In this section, we collect some relevant facts about Lie algebroids. More systematic treatments of them can be found in [Mc21] [H-M] and [X-M].

A **Lie algebroid** over a smooth manifold P is a vector bundle A over P together with a Lie algebra structure on the space $\Gamma(A)$ of smooth sections of A and a bundle map $\rho : A \rightarrow TP$ (called the anchor map of the Lie algebroid) such that

1) ρ defines a Lie algebra homomorphism from $\Gamma(A)$ to the space $\chi(P)$ of vector fields with the commutator Lie algebra structure, and

2) for $f \in C^\infty(P)$, $\omega_1, \omega_2 \in \Gamma(A)$ the following derivation law holds:

$$\{\omega_1, f\omega_2\} = f\{\omega_1, \omega_2\} + (\rho(\omega_1)f)\omega_2.$$

Immediate examples of Lie algebroids are 1) the tangent bundle TP as a Lie algebroid over P with the identity map as the anchor map, and 2) a Lie algebra \mathfrak{g} as a Lie algebroid over a one point space.

Transformation Lie algebroids. Let $\sigma : G \times P \rightarrow P$ be an action of a Lie group G on a manifold P . For each $x \in \mathfrak{g}$, the Lie algebra of G , denote by σ_x the vector field on P defined by

$$\sigma_x(p) = \left. \frac{d}{dt} \right|_{t=0} \sigma(\exp tx, p), \quad p \in P.$$

Then there is a natural Lie algebroid structure on the trivial vector bundle $P \times \mathfrak{g}$ with $-\sigma$ as the anchor map. Here we regard $\sigma : x \mapsto \sigma_x$ as a bundle map from $P \times \mathfrak{g}$ to TP . The Lie bracket on the space $\Gamma(P \times \mathfrak{g}) \cong C^\infty(P, \mathfrak{g})$ of smooth sections of $P \times \mathfrak{g}$ is given by

$$\{\bar{x}, \bar{y}\} = [\bar{x}, \bar{y}]_{\mathfrak{g}} - \sigma_{\bar{x}} \cdot \bar{y} + \sigma_{\bar{y}} \cdot \bar{x}, \quad (1)$$

where the first term on the right hand side denotes the pointwise Lie bracket in \mathfrak{g} , and the second term denotes the derivative of the \mathfrak{g} -valued function \bar{y} in the direction of the vector field $\sigma_{\bar{x}}$.

Cotangent bundle Lie algebroids. Let (P, π) be a Poisson manifold with Poisson bivector field π . We use $\pi^\#$ to denote the bundle map

$$\pi^\#(p) : T_p^*P \rightarrow T_pP : \alpha_p \mapsto -\alpha_p \lrcorner \pi(p),$$

or

$$(\beta_p, \pi^\#(\alpha_p)) = \pi(p)(\beta_p, \alpha_p), \quad \alpha_p, \beta_p \in T_p^*P.$$

(Note the sign convention here). Then, with $-\pi^\#$ as the anchor map, the cotangent bundle T^*P becomes a Lie algebroid over P , where the Lie bracket on the space $\Omega^1(P)$ of 1-forms on P is given by

$$\{\alpha, \beta\} = d\pi(\alpha, \beta) - \pi^\# \alpha \lrcorner d\beta + \pi^\# \beta \lrcorner d\alpha \quad (2)$$

$$= -d\pi(\alpha, \beta) - L_{\pi^\# \alpha} \beta + L_{\pi^\# \beta} \alpha, \quad (3)$$

where $L_{\pi^\# \alpha} \beta$ denotes the Lie derivative of the 1-form β in the direction of the vector field $\pi^\# \alpha$.

Lie algebroid morphisms. The definition of Lie algebroid morphisms between Lie algebroids over different bases is rather involved [H-M] [X-M], the reason being that a bundle map does not necessarily induce a map between sections. We will only need the definition for the following special case. Let A be a Lie algebroid over P with anchor map ρ . Let \mathfrak{g} be a Lie algebra, considered as a Lie algebroid over a one point space. A smooth map

$$\phi : A \longrightarrow \mathfrak{g}$$

which is linear on each fiber is said to be a Lie algebroid morphism if for any sections ω_1 and ω_2 of A ,

$$\phi\{\omega_1, \omega_2\} = [\phi(\omega_1), \phi(\omega_2)]_{\mathfrak{g}} + \rho(\omega_1) \cdot \phi(\omega_2) - \rho(\omega_2) \cdot \phi(\omega_1),$$

where the first term on the right hand side denotes the pointwise Lie bracket in \mathfrak{g} , and the second term denotes the Lie derivative of the \mathfrak{g} -valued function $\phi(\omega_2)$ in the direction of the vector field $\rho(\omega_1)$.

Example 2.1 In the case of a transformation Lie algebroid $A = P \times \mathfrak{g}$, the projection map $P \times \mathfrak{g} \rightarrow \mathfrak{g}$ is a Lie algebroid morphism.

Representations of Lie algebroids. Let A be a Lie algebroid over P with anchor map ρ . Let E be a vector bundle over P . A representation of A on E is a k -linear map

$$\Gamma(A) \otimes \Gamma(E) \longrightarrow \Gamma(E) : a \otimes s \longmapsto D_a s,$$

where $\Gamma(A)$ and $\Gamma(E)$ denote respectively the spaces of smooth sections of A and E , such that for any $a, b \in \Gamma(A)$, $s \in \Gamma(E)$ and $f \in C^\infty(P)$,

$$\begin{aligned} (1) \quad D_{fa}(s) &= fD_as; \\ (2) \quad D_a(fs) &= fD_as + (\rho(a)f)s; \\ (3) \quad D_a(D_bs) - D_b(D_as) &= D_{\{a,b\}}s. \end{aligned}$$

If $\langle \cdot, \cdot \rangle$ is a smooth section of $S^2(E^*)$, the second symmetric power of the dual bundle of E , we say that the representation of A on E is $\langle \cdot, \cdot \rangle$ -preserving if for any $a \in \Gamma(A)$, $s_1, s_2 \in \Gamma(E)$,

$$\rho(a)(s_1, s_2) = \langle D_as_1, s_2 \rangle + \langle s_1, D_as_2 \rangle.$$

Example 2.2 A representation of the tangent bundle TP over P is a vector bundle E over P together with a flat linear connection.

Example 2.3 Any transformation Lie algebroid $P \times \mathfrak{g}$ has a natural representation on the tangent bundle TP of P via

$$D_{\bar{x}}V = -[\sigma_{\bar{x}}, V] - \sigma_{V \cdot \bar{x}}, \quad (4)$$

where V is a vector field on P , \bar{x} is a section of $P \times \mathfrak{g}$, considered as a \mathfrak{g} -valued function of P , and $V \cdot \bar{x}$ denotes the Lie derivative of \bar{x} in the direction of V . Correspondingly, there a representation of $P \times \mathfrak{g}$ on the cotangent bundle T^*P satisfying

$$-\sigma_{\bar{x}}(\alpha, V) = (D_{\bar{x}}\alpha, V) + (\alpha, D_{\bar{x}}V) \quad (5)$$

for any 1-form α on P . Or, equivalently,

$$(D_{\bar{x}}\alpha, V) = -\sigma_{\bar{x}}(\alpha, V) + (\alpha, [\sigma_{\bar{x}}, V] + \sigma_{V \cdot \bar{x}}). \quad (6)$$

Notice that when \bar{x} is a constant section of $P \times \mathfrak{g}$ corresponding to $x \in \mathfrak{g}$, the 1-form $D_x\alpha$ is given by the Lie derivative

$$D_x\alpha = -L_{\sigma_x}\alpha.$$

Example 2.4 Let A be a Lie algebroid over P with anchor map ρ . Suppose that $\phi : A \rightarrow \mathfrak{g}$ is a Lie algebroid morphism from A to a Lie algebra \mathfrak{g} . Then for any vector space U with a \mathfrak{g} -action, there is a representation of A on the trivial vector bundle $P \times U$ given by

$$D_a \bar{u} = \rho(a) \cdot \bar{u} + \phi(a)(\bar{u}), \quad (7)$$

where a is a section of A , \bar{u} is a section of $P \times U$, and $\phi(a)(\bar{u})$ denotes the action of $\phi(a) \in C^\infty(P, \mathfrak{g})$ on $\bar{u} \in C^\infty(P, U)$ taken pointwise over P . In particular, there is a representation of A on the trivial vector bundle $P \times \mathfrak{g}^*$ given by

$$D_a \bar{\xi} = \rho(a) \cdot \bar{\xi} + ad_{\phi(a)}^* \bar{\xi},$$

where $\bar{\xi} \in C^\infty(P, \mathfrak{g}^*)$ is a section of $P \times \mathfrak{g}^*$, and the co-adjoint representation of \mathfrak{g} on \mathfrak{g}^* is defined by

$$(ad_x^* \xi, y) = -(\xi, [x, y]), \quad x, y \in \mathfrak{g}, \xi \in \mathfrak{g}^*.$$

Lie algebroid cohomology Let A be a Lie algebroid over P with anchor map ρ . Let E be a representation of A . Define

$$d : \Gamma(\wedge^{k-1} A^* \otimes E) \longrightarrow \Gamma(\wedge^k A^* \otimes E), \quad k = 1, 2, 3, \dots$$

by

$$df(a_1, \dots, a_k) = \sum_i (-1)^{i+1} D_{a_i} f(a_1, \dots, \hat{a}_i, \dots, a_k) + \sum_{i < j} (-1)^{i+j} f(\{a_i, a_j\}, \dots, \hat{a}_i, \dots, \hat{a}_j, \dots, a_k). \quad (8)$$

Then d is well-defined and that $d^2 = 0$. The cohomology of $(C^\bullet(A, E), d)$ is called the Lie algebroid cohomology of A with coefficients in E , and it is denoted by $H^\bullet(A, E)$. Elements of $H^0(A, E)$ are called A -parallel sections of E . When E is the trivial 1-dimensional vector bundle with the representation of A given by

$$D_a f = \rho(a)f, \quad f \in C^\infty(P),$$

the cohomology of A with coefficients in E is called the cohomology of A with trivial coefficients.

In the case when $A = TP$ is the tangent bundle of P , the Lie algebroid cohomology of A with trivial coefficients is nothing but the de Rham cohomology of P .

In the case when $A = T^*P$ is the cotangent bundle Lie algebroid of a Poisson manifold (P, π) , the Lie algebroid cohomology of A with trivial coefficients is called the Poisson cohomology of (P, π) . The cochain complex is the space $\Gamma(\wedge^\bullet TP)$ of multi-vector fields on P , and the coboundary operator d can be expressed as $d = [\pi, \bullet]$, where $[\ , \]$ stands for the Schouten bracket [Ku] on the multi-vector fields on P [Li]

In the case when $A = P \times \mathfrak{g}$ is a transformation Lie algebroid, the Lie algebra cohomology of A with trivial coefficients is simply the Lie algebra cohomology of \mathfrak{g} with coefficients in $C^\infty(P)$.

Transitive Lie algebroids. A Lie algebroid A over a manifold P with anchor map $\rho : A \rightarrow TP$ is said to be transitive if ρ has full rank at every point of P . In this case, the subbundle L of A defined by the kernel of ρ is a Lie algebra bundle in the sense that each fiber has a Lie algebra structure and local trivializations exist [Mc1]. Any section $\gamma : TP \rightarrow A$ of ρ defines a linear connection ∇ of L by

$$\nabla_V^\gamma(l) = \{\gamma(V), l\}, \quad V \in \chi(P), \quad l \in \Gamma(L),$$

and it satisfies

$$\nabla_V^\gamma[l_1, l_2] = [\nabla_V^\gamma l_1, l_2] + [l_1, \nabla_V^\gamma l_2], \quad \forall l_1, l_2 \in \Gamma(L). \quad (9)$$

3 Poisson Lie groups and Poisson actions

In this section, we collect some facts about Poisson Lie groups that will be used in this paper. Details can be found in [Dr1] [STS] [KS] and [Lu1].

Assume that (G, π_G) is a Poisson Lie group with Poisson bi-vector field π_G . For each $g \in G$, the element $r_{g^{-1}}\pi_G(g)$ is in $\wedge^2 \mathfrak{g}$, where $r_{g^{-1}}$ denotes the right translation on $\wedge^2 TG$ by g^{-1} . The derivative of the map

$$G \longrightarrow \wedge^2 \mathfrak{g} : g \longmapsto r_{g^{-1}}\pi_G(g)$$

at the identity element e of G is denoted by δ , so we have

$$\delta : \mathfrak{g} \longrightarrow \wedge^2 \mathfrak{g} : \delta(x) = \left. \frac{d}{dt} \right|_{t=0} r_{\exp(-tx)} \pi_G(\exp tx). \quad (10)$$

The dual map of δ defines a Lie bracket on \mathfrak{g}^* , i.e.,

$$([\xi, \eta], x) = (\delta(x), \xi \wedge \eta), \quad x \in \mathfrak{g}, \quad \xi, \eta \in \mathfrak{g}^*. \quad (11)$$

The pair (\mathfrak{g}, δ) (or the pair $(\mathfrak{g}, \mathfrak{g}^*)$) is called the tangent Lie bialgebra of the Poisson Lie group G [Dr1]. For $x \in \mathfrak{g}$ and $\xi \in \mathfrak{g}^*$, denote by $ad_x^* \xi \in \mathfrak{g}^*$ and $ad_\xi^* x \in \mathfrak{g}$ respectively the elements given by

$$\begin{aligned}\langle ad_x^* \xi, y \rangle &= -\langle \xi, [x, y] \rangle \quad \forall y \in \mathfrak{g} \\ \langle ad_\xi^* x, \eta \rangle &= -\langle x, [\xi, \eta] \rangle \quad \forall \eta \in \mathfrak{g}^*.\end{aligned}$$

The double Lie algebra \mathfrak{d} is the vector space $\mathfrak{g} \oplus \mathfrak{g}^*$ with the Lie bracket

$$[x_1 + \xi_1, x_2 + \xi_2] = [x_1, x_2] + ad_{\xi_1}^* x_2 - ad_{\xi_2}^* x_1 + [\xi_1, \xi_2] + ad_{x_1}^* \xi_2 - ad_{x_2}^* \xi_1. \quad (12)$$

It is the unique Lie bracket on the vector space $\mathfrak{g} \oplus \mathfrak{g}^*$ characterized by the properties that 1) when restricted to \mathfrak{g} and \mathfrak{g}^* , it coincides with the given Lie brackets on \mathfrak{g} and \mathfrak{g}^* , and 2) the scalar product $\langle \cdot, \cdot \rangle_{\mathfrak{d}}$ on $\mathfrak{g} \oplus \mathfrak{g}^*$ defined by

$$\langle x_1 + \xi_1, x_2 + \xi_2 \rangle_{\mathfrak{d}} = (x_1, \xi_2) + (\xi_1, x_2), \quad x_1, x_2 \in \mathfrak{g}, \xi_1, \xi_2 \in \mathfrak{g}^* \quad (13)$$

is ad-invariant with respect to this Lie bracket. We use $\mathfrak{d} = \mathfrak{g} \bowtie \mathfrak{g}^*$ to denote this Lie algebra, indicating the fact that the Lie bracket is built out of the Lie brackets on \mathfrak{g} and on \mathfrak{g}^* and their co-adjoint actions on each other.

Let G^* be the simply-connected Lie group with Lie algebra \mathfrak{g}^* . Then it is also a Poisson Lie group with tangent Lie bialgebra $(\mathfrak{g}^*, \mathfrak{g})$. It is called the dual Poisson Lie group of G .

Assume that D is a connected Lie group with Lie algebra \mathfrak{d} . Assume also that both G and G^* can be embedded in D as subgroups corresponding to the Lie algebra inclusions $\mathfrak{g} \subset \mathfrak{d}$ and $\mathfrak{g}^* \subset \mathfrak{d}$. Assume furthermore that every element of D can be uniquely written as a product gu for some $g \in G$ and $u \in G^*$. Then we can identify G with the quotient space D/G^* . Consequently, there is a left action of D on G by left translations. We will denote it by

$$D \times G \longrightarrow G : (d, g) \longmapsto {}^d g. \quad (14)$$

Infinitesimally, this defines a Lie algebra anti-homomorphism from \mathfrak{d} to the Lie algebra $\chi(G)$ of vector fields on G :

$$\mathfrak{d} \longrightarrow \chi(G) : x + \xi \longmapsto \rho_{x+\xi}(g) = r_g x - r_g \left(\xi \lrcorner (r_{g^{-1}} \pi_G(g)) \right), \quad g \in G. \quad (15)$$

This action of \mathfrak{d} on G will be used in Example 5.2. The vector fields $-\rho_\xi$, $\xi \in \mathfrak{g}^*$, are called the right dressing vector fields on G [Lu1]. When the global decomposition $D = GG^*$ does not hold, we will assume that these vector fields integrate to a left action of D on G .

Similarly, by identifying G with the quotient space $G^* \backslash D$, we get a right action of D on G which we denote by

$$G \times D \longrightarrow G : (g, d) \longmapsto g^d. \quad (16)$$

The corresponding infinitesimal right action of D on G is given by

$$\mathfrak{d} \longrightarrow \chi(G) : x + \xi \longmapsto \lambda_{x+\xi}(g) = l_g x + r_g(Ad_{g^{-1}}^* \xi \lrcorner (r_{g^{-1}} \pi_G(g))). \quad (17)$$

This action will be used in Theorem 7.2. The vector fields $-\lambda_\xi$, for $\xi \in \mathfrak{g}^*$, are called the left dressing vector fields on G [Lu1].

On the other hand, the group G acts on \mathfrak{d} by Lie algebra automorphisms via

$$Ad_g(x + \xi) = Ad_g x + (Ad_{g^{-1}}^* \xi \lrcorner (r_{g^{-1}} \pi_G(g)) + Ad_{g^{-1}}^* \xi. \quad (18)$$

The corresponding action of \mathfrak{g} on \mathfrak{d} is simply the adjoint action of \mathfrak{d} on itself restricted to \mathfrak{g} . This action will be used in Proposition 4.4. Similarly, the group G^* acts on \mathfrak{d} by Lie algebra automorphisms via

$$Ad_u(x + \xi) = Ad_{u^{-1}}^* x + Ad_u \xi + (Ad_{u^{-1}}^* x \lrcorner (r_{u^{-1}} \pi_{G^*}(u))). \quad (19)$$

This action will be used in Examples 4.7 and 4.8.

We now turn to Poisson actions by G . For any action $\sigma : G \times P \rightarrow P$ of G on a manifold P , we can define

$$\phi : T^*P \longrightarrow \mathfrak{g}^* : (\phi(\alpha_p), x) = (\alpha_p, \sigma_x(p)), \quad x \in \mathfrak{g}, \alpha_p \in T_p^*P, p \in P. \quad (20)$$

Each 1-form α on P then defines a \mathfrak{g}^* -valued function $\phi(\alpha)$ on P :

$$\phi(\alpha)(p) = \phi(\alpha_p).$$

Assume now that (P, π) is a Poisson manifold and G is a Poisson Lie group. Recall that an action $\sigma : G \times P \rightarrow P$ is said to be Poisson if for any $g \in G, p \in P$

$$\pi(gp) = g_* \pi(p) + p_* \pi_G(g),$$

where p_* is the differential of the map from G to P given by $g \mapsto gp$. When G is connected, this is equivalent to the following infinitesimal criterion [Lu-We]:

$$L_{\sigma_{\bar{x}}} \pi = \sigma_{\delta(x)} \quad (21)$$

for all $x \in \mathfrak{g}$, where $\delta(x) \in \mathfrak{g} \wedge \mathfrak{g}$ is given by (10), and $\sigma_{\delta(x)} = \sum_i \sigma_{x_i} \wedge \sigma_{y_i}$ if $\delta(x) = \sum_i x_i \wedge y_i$. This is then equivalent to

$$\phi(\{\alpha, \beta\}) = [\phi(\alpha), \phi(\beta)]_{\mathfrak{g}^*} - \pi^\#(\alpha) \cdot \phi(\beta) + \pi^\#(\beta) \cdot \phi(\alpha) \quad (22)$$

for any 1-forms α and β on P . In other words, we have the following Proposition:

Proposition 3.1 *[Xu] Assume that G is connected. The action σ is a Poisson action if and only if the map*

$$\phi : T^*P \longrightarrow \mathfrak{g}^*$$

*defined by (20) is a Lie algebroid morphism, where T^*P has the cotangent bundle Lie algebroid structure defined by the Poisson structure π on P , and \mathfrak{g}^* has the Lie algebra structure given by (11), considered as a Lie algebroid over a one point space.*

4 The Lie algebroid $A = (P \times \mathfrak{g}) \bowtie T^*P$

In this section, we describe, for every Poisson manifold P with a Poisson action $\sigma : G \times P \rightarrow P$ of a Poisson Lie group G , a Lie algebroid structure over P on the direct sum vector bundle $(P \times \mathfrak{g}) \oplus T^*P$.

First, the action of G on P defines a representation of the transformation Lie algebroid $P \times \mathfrak{g}$ on T^*P given by (6) as in Example 2.3.

Secondly, using the Lie groupoid homomorphism

$$\phi : T^*P \longrightarrow \mathfrak{g}^*$$

in Proposition 3.1, we get (see Example 2.4) a representation of the cotangent bundle Lie algebroid T^*P on the trivial vector bundle $P \times \mathfrak{g}$ induced from the co-adjoint representation of \mathfrak{g}^* on \mathfrak{g} . It is given by

$$D_\alpha \bar{x} = -\pi^\#(\alpha) \cdot \bar{x} + ad_{\phi(\alpha)}^* \bar{x} \quad (23)$$

where $(ad_{\phi(\alpha)}^* \bar{x})(p) = ad_{\phi(\alpha)(p)}^* \bar{x}(p)$ for $p \in P$.

Theorem 4.1 *Let $\sigma : G \times P \rightarrow P$ be a Poisson action of the Poisson Lie group G on the Poisson manifold P . Let*

$$A \stackrel{def}{=} (P \times \mathfrak{g}) \oplus T^*P \quad (24)$$

be the direct sum vector bundle over P . Then there is a Lie algebroid structure on A such that both the transformation Lie algebroid $(P \times \mathfrak{g}, -\sigma)$ and the cotangent bundle Lie algebroid $(T^*P, -\pi^\#)$ are Lie subalgebroids of A . The anchor map of A is $-\sigma - \pi^\#$. The Lie bracket between a section \bar{x} of $P \times \mathfrak{g}$ and a section α of T^*P is given by

$$\{\bar{x}, \alpha\} = -D_\alpha \bar{x} + D_{\bar{x}} \alpha, \quad (25)$$

where $D_\alpha \bar{x}$ is given by (23) and $D_{\bar{x}} \alpha$ by (6);

Definition 4.2 We use $A = (P \times \mathfrak{g}) \bowtie T^*P$ to denote this Lie algebroid, indicating the fact that the Lie bracket on the sections of A uses both the representation of $P \times \mathfrak{g}$ on T^*P and the representation of T^*P on $P \times \mathfrak{g}$.

Examples will be given in Section 5. The proof of Theorem 4.1 will be given in Section 6. We now discuss some properties of the Lie algebroid A .

By considering each element of \mathfrak{g} as a constant section of A , we get a Lie algebra homomorphism

$$i : \mathfrak{g} \longrightarrow \Gamma(A) : i(x)_p = x, \quad \forall p \in P. \quad (26)$$

It induces an action of \mathfrak{g} on $\Gamma(A)$ by

$$\mathfrak{g} \ni x : (\bar{y}, \beta) \longmapsto \{i(x), (\bar{y}, \beta)\} = ([x, \bar{y}]_{\mathfrak{g}} - \sigma_x \cdot \bar{y} - ad_{\phi(\beta)}^* x, -L_{\sigma_x} \beta), \quad (27)$$

where $\bar{y} \in C^\infty(P, \mathfrak{g})$ and β is a 1-form on P .

Proposition 4.3 The following defines a left action of G on A :

$$g \cdot (x, \alpha_p) = (Ad_g x + Ad_{g^{-1}}^* \phi(\alpha_p) \lrcorner (r_{g^{-1}} \pi_G(g)), (g^{-1})^* \alpha_p), \quad g \in G, p \in P, \quad (28)$$

where $x \in \mathfrak{g}, \alpha_p \in T_p^*P$. It has the following properties:

(1) The action of \mathfrak{g} on $\Gamma(A)$ induced by this G -action on A coincides with the action given by (27).

(2) The action of G commutes with the anchor map $-\sigma - \pi^\#$ of A , and G acts on $\Gamma(A)$ as automorphisms with respect to the Lie bracket $\{ , \}$.

(3) For any $x \in \mathfrak{g}$ and $g \in G$, $i(Ad_g x) = g \cdot i(x)$.

In other words, A is a Harish-Chandra G -Lie algebroid [B-B].

(4) The action of G preserves the scalar product on A given by

$$\langle x + \alpha_p, y + \beta_p \rangle_p = (\beta_p, \sigma_x(p)) + (\alpha_p, \sigma_y(p)) = \phi(\beta_p)(x) + \phi(\alpha_p)(y). \quad (29)$$

Proof. The proof is straightforward. We just point out that to prove that (28) defines an action, one only needs the fact that the Poisson structure π_G on G makes G into a Poisson Lie group, i.e.,

$$\pi_G(gh) = l_g\pi_G(h) + r_h\pi_G(g), \quad \forall g, h \in G.$$

In the proof of (1), one needs the fact that $\pi^\#$ vanishes at the identity point of G . (2) is a reformulation of the fact that σ is a Poisson action. (3) is obvious. (4) is straightforward.

Q.E.D.

Proposition 4.4 1) *the map*

$$\Phi : A = (P \times \mathfrak{g}) \bowtie T^*P \longrightarrow \mathfrak{d} : (x, \alpha_p) \longmapsto x + \phi(\alpha_p) \quad (30)$$

is a Lie algebroid morphism from A to the Lie algebra $\mathfrak{d} = \mathfrak{g} \bowtie \mathfrak{g}^$, considered as a Lie algebroid over a one point space. With respect to the following action of G on \mathfrak{d} (see (18) in Section 3:*

$$Ad_g(x + \xi) = Ad_g x + Ad_{g^{-1}}^* \xi \lrcorner (r_{g^{-1}}\pi_G(g)) + Ad_{g^{-1}}^* \xi,$$

where $g \in G, x \in \mathfrak{g}$ and $\xi \in \mathfrak{g}^$, the map Φ is in fact a morphism of Harish-Chandra G -Lie algebroids;*

2) *The map Φ induces the following representation of A on the trivial vector bundle $P \times \mathfrak{d}$ via the adjoint representation of \mathfrak{d} on itself:*

$$D_{\bar{x}+\alpha}^A(\bar{y} + \bar{\xi}) = (-\sigma_{\bar{x}} - \pi^\# \alpha) \cdot (\bar{y} + \bar{\xi}) + [\bar{x} + \phi(\alpha), \bar{y} + \bar{\xi}]_{\mathfrak{d}} \quad (31)$$

where $\bar{y} + \bar{\xi}$ is a section of $P \times \mathfrak{d}$ with $\bar{y} \in C^\infty(P, \mathfrak{g})$ and $\bar{\xi} \in C^\infty(P, \mathfrak{g}^)$. This representation of A on $P \times \mathfrak{d}$ preserves the symmetric product $\langle \cdot, \cdot \rangle_{\mathfrak{d}}$ on $P \times \mathfrak{d}$ given by (13) on each fiber.*

Proof. 1) is easily reduced to the fact that for any $\bar{x} \in C^\infty(P, \mathfrak{g})$ and any 1-form α on P ,

$$\phi(D_{\bar{x}}\alpha) = -\sigma_{\bar{x}} \cdot \phi(\alpha) + ad_{\bar{x}}^* \phi(\alpha). \quad (32)$$

This identity can be proved directly from the definition of $D_{\bar{x}}\alpha$. If we equip the trivial vector bundle $P \times \mathfrak{g}^*$ the representation of the transformation Lie algebroid $P \times \mathfrak{g}$ given by

$$D_{\bar{x}}\bar{\xi} = -\sigma_{\bar{x}} \cdot \bar{\xi} + ad_{\bar{x}}^* \bar{\xi}.$$

Then (32) is saying that the bundle map

$$T^*P \longrightarrow P \times \mathfrak{g}^* : \alpha_p \longmapsto (p, \phi(\alpha_p))$$

is a $(P \times \mathfrak{g})$ -morphism. Proof of 2) is straightforward.

Q.E.D.

In general, if A is any Lie algebroid over P with anchor map ρ , the kernel ρ in each fiber A_p has an induced Lie algebra structure, namely, for any $a_p, b_p \in \ker(\rho|_{A_p})$, extend them arbitrarily to sections a and b of A and define

$$[a_p, b_p] = \{a, b\}(p).$$

This is independent of the extensions.

For a transformation Lie algebroid $P \times \mathfrak{g}$, the kernels of the anchor map $-\sigma$ are the stabilizer Lie subalgebras

$$\mathfrak{g}_p = \{x \in \mathfrak{g} : \sigma_x(p) = 0\}, \quad (33)$$

For the cotangent bundle Lie algebroid T^*P of a Poisson manifold (P, π) , we get the Lie algebra

$$\mathfrak{t}_p = \{\alpha_p \in T_p^*P : \pi^\#(\alpha_p) = 0\} \quad (34)$$

at each point $p \in P$. They are called the transversal Lie algebras to the symplectic leaves of the Poisson structure on P [We].

For the Lie algebroid $A = (P \times \mathfrak{g}) \bowtie T^*P$ constructed in Theorem 4.1, the kernel of the anchor map $-\sigma - \pi^\#$ in the fiber A_p over $p \in P$ is the space

$$\mathfrak{l}_p = \{(x, \alpha_p) : x \in \mathfrak{g}, \alpha_p \in T_p^*P : \sigma_x(p) + \pi^\#(\alpha_p) = 0\}. \quad (35)$$

It is an isotropic subspace of A_p with respect to $\langle \cdot, \cdot \rangle_p$ given by (29).

Proposition 4.5 (1) For each $p \in P$, the map

$$\Phi : \mathfrak{l}_p \longrightarrow \mathfrak{d} = \mathfrak{g} \bowtie \mathfrak{g}^* : (x, \alpha_p) \longmapsto x + \phi(\alpha_p) \quad (36)$$

is a Lie algebra homomorphism from \mathfrak{l}_p to the double Lie algebra \mathfrak{d} of $(\mathfrak{g}, \mathfrak{g}^*)$.

(2) For any $g \in G$ and $p \in P$, we have $g \cdot \mathfrak{l}_p = \mathfrak{l}_{gp}$, where $g \cdot \mathfrak{l}_p$ denotes the image of \mathfrak{l}_p in A_{gp} under the action of g on A given by (28).

The proof of this Proposition is again straightforward.

Proposition 4.6 *When P is symplectic or when P is G -homogeneous, the Lie algebroid $A = (P \times \mathfrak{g}) \bowtie T^*P$ is transitive. Consequently, the subbundle L of A defined by the kernel of the anchor map $-\sigma - \pi^\#$ is a Lie algebra bundle of rank $= \dim \mathfrak{g}$.*

In general, the dimensions of the Lie algebra \mathfrak{l}_p 's may vary. Consider again the stabilizer Lie subalgebra \mathfrak{g}_p and the transversal Lie algebra \mathfrak{t}_p at each point $p \in P$. Clearly,

$$\mathfrak{g}_p \subset \mathfrak{l}_p, \quad \mathfrak{t}_p \subset \mathfrak{l}_p$$

as Lie subalgebras, and

$$\mathfrak{g}_p \cap \mathfrak{t}_p = 0.$$

Set

$$O_p = \{\sigma_x(p) : x \in \mathfrak{g}\} \subset T_p P, \quad (37)$$

$$S_p = \{\pi^\#(\alpha_p) : \alpha_p \in T_p^* P\} \subset T_p P. \quad (38)$$

Then

$$\begin{aligned} \dim \mathfrak{l}_p &= \dim \mathfrak{g} + \dim P - \dim(O_p + S_p) \\ &= (\dim \mathfrak{g} - \dim O_p) + (\dim P - \dim S_p) + \dim(O_p \cap S_p) \\ &= \dim \mathfrak{g}_p + \dim \mathfrak{t}_p + \dim(O_p \cap S_p). \end{aligned}$$

Example 4.7 When P is symplectic, i.e., when $\pi^\# : T_p^* P \rightarrow T_p P$ is one-to-one for each $p \in P$, the map $\Phi : \mathfrak{l}_p \rightarrow \mathfrak{d}$ is injective, so $\Phi(\mathfrak{l}_p)$ is a maximal isotopic Lie subalgebra of \mathfrak{d} .

Let ω be the symplectic 2-form on P related to π by

$$\omega(\pi^\# \alpha, \pi^\# \beta) = \pi(\alpha, \beta).$$

Then

$$\Phi(\mathfrak{l}_p) = \{x - x \lrcorner \Phi(\omega_p) : x \in \mathfrak{g}\} \subset \mathfrak{d},$$

where $\Phi(\omega_p) = (\Phi \wedge \Phi)(\omega_p) \in \mathfrak{g}^* \wedge \mathfrak{g}^*$. Notice that $\Phi(\mathfrak{l}_p)$ is transversal to \mathfrak{g}^* in \mathfrak{d} . Thus $(\mathfrak{d}, \mathfrak{g}^*, \Phi(\mathfrak{l}_p))$ form a Manin triple [Dr1], and $(\mathfrak{g}^*, \Phi(\mathfrak{l}_p))$ becomes a Lie bialgebra.

Let (G^*, π_{G^*}) be the simply-connected Poisson Lie group dual to G . The tangent Lie bialgebra of G^* is $(\mathfrak{g}^*, \mathfrak{g})$. Define

$$\pi_{G^*,p} = \pi_{G^*} + (\Phi(\omega_p))^r - (\Phi(\omega_p))^l,$$

where $(\Phi(\omega_p))^r$ (resp. $(\Phi(\omega_p))^l$) is the right invariant bi-vector field on G^* with value $\Phi(\omega_p) \in \mathfrak{g}^* \wedge \mathfrak{g}^*$ at the identity element $e \in G^*$. Then $(G^*, \pi_{G^*,p})$ is also a Poisson Lie group, and its tangent Lie bialgebra is $(\mathfrak{g}^*, \Phi(\mathfrak{l}_p))$ [Lu1] [Da-So].

The bi-vector field $\pi_{G^*} + (\Phi(\omega_p))^r$ is also Poisson. It is an example of an affine Poisson structure on G^* [Lu1] [Da-So]. When P is connected and simply-connected, it is shown in [Lu1] that for any $p \in P$ there exists a unique map $J_p : P \rightarrow G^*$ such that

- (1) $J_p(p) = e$;
- (2) $\pi^\#(J_p^* x^l) = \sigma_x$ for every $x \in \mathfrak{g}$, where x^l is the left invariant 1-form on G^* such that $x^l(e) = x$;
- (3) $J_p : (P, \pi) \rightarrow (G^*, \pi_{G^*} + (\Phi(\omega_p))^r)$ is a Poisson map.

The map J_p is called a moment map for the Poisson action σ of G on P [Lu2].

If p_1 is any other point in P , the maps J_p and J_{p_1} are related by

$$J_p(q) = J_p(p_1) J_{p_1}(q), \quad q \in P,$$

and

$$\Phi(\omega_p) = -r_{u^{-1}} \pi_{G^*}(u) + Ad_u \Phi(\omega_{p_1}).$$

The Manin triples $(\mathfrak{d}, \mathfrak{g}^*, \Phi(\mathfrak{l}_p))$ and $(\mathfrak{d}, \mathfrak{g}^*, \Phi(\mathfrak{l}_{p_1}))$ are related by

$$(\mathfrak{d}, \mathfrak{g}^*, \Phi(\mathfrak{l}_p)) = Ad_{J_p(p_1)} (\mathfrak{d}, \mathfrak{g}^*, \Phi(\mathfrak{l}_{p_1}))$$

where $J_p(p_1) \in G^*$ acts on \mathfrak{d} by (19).

Example 4.8 The left dressing vector fields on G^* are defined by

$$\sigma_x(u) = \pi_{G^*}^\#(x^l) \tag{39}$$

where $x \in \mathfrak{g}$. The map

$$\mathfrak{g} \longrightarrow \chi(G^*) : x \longmapsto \sigma_x$$

defines a Lie algebra anti-homomorphism from \mathfrak{g} to the Lie algebra $\chi(G^*)$ of vector fields on G^* . Assume that these vector fields integrate to an action of G on G^* . It is called the

left dressing action of G on G^* [STS]. It is a Poisson action. It in fact has the identity map $G^* \rightarrow G^*$ as a moment map [Lu2]. The orbits of the dressing action are exactly the symplectic leaves of G^* . Let P be such an orbit. Then the action $G \times P \rightarrow P$ of G on P is a Poisson action.

For $u \in P \subset G^*$, the Lie algebra \mathfrak{l}_u can be identified with the Lie subalgebra $Ad_{u^{-1}}\mathfrak{g}$ of \mathfrak{d} , where, again, $Ad_{u^{-1}} : \mathfrak{d} \rightarrow \mathfrak{d}$ denotes the action of $u^{-1} \in G^*$ on \mathfrak{d} given by (19).

Example 4.9 When P is a homogeneous G -space, we have $\dim \mathfrak{l}_p = \dim \mathfrak{g}$ for each $p \in P$, and the map $\Phi : \mathfrak{l}_p \rightarrow \mathfrak{d}$ is injective. Therefore, each $\Phi(\mathfrak{l}_p)$ is a maximal isotropic Lie subalgebra of \mathfrak{d} . They are used in [Dr2] to classify Poisson homogeneous G -spaces. We will treat them in more details in Section 7.

Example 4.10 When the Poisson bi-vector field π on P vanishes at a certain point $p \in P$, we have

$$\mathfrak{l}_p = \{(x, \alpha_p) : x \in \mathfrak{g}, \alpha_p \in T_p^*P, \sigma_x(p) = 0\} \cong \mathfrak{g}_p \oplus T_p^*P.$$

To describe the Lie algebra structure on \mathfrak{l}_p , we first note that the action of G_p on P , where G_p is the stabilizer subgroup of G at p , linearizes to an action of G_p on T_pP and thus on T_p^*P . Denote the corresponding action of \mathfrak{g}_p on T_p^*P by

$$\mathfrak{g}_p \times T_p^*P \longrightarrow T_p^*P : (x, \alpha_p) \longmapsto x \cdot \alpha_p. \quad (40)$$

On the other hand, since the action is Poisson, we know from (21) that

$$\sigma_{\delta(x)}(p) = 0, \quad \forall x \in \mathfrak{g}_p.$$

Thus

$$(x, [\phi(\alpha_p), \phi(\beta_p)]_{\mathfrak{g}^*}) = 0, \quad \forall x \in \mathfrak{g}_p, \alpha_p, \beta_p \in T_p^*P.$$

In other words,

$$ad_{\phi(\alpha_p)}^* \mathfrak{g}_p \subset \mathfrak{g}_p, \quad \forall \alpha_p \in T_p^*P.$$

The Lie algebra structure on \mathfrak{l}_p is now given by

$$[(x, \alpha_p), (y, \beta_p)] = ([x, y] + ad_{\phi(\alpha_p)}^* y - ad_{\phi(\beta_p)}^* x, [\alpha_p, \beta_p] + x \cdot \beta_p - y \cdot \alpha_p). \quad (41)$$

Note that the Lie algebra $\mathfrak{l}_p = \mathfrak{g}_p + T_p^*P$ is an example of a double Lie algebra [Lu-We] [KS] (called a matched pair of Lie algebras in [Mj]) in the sense that it contains \mathfrak{g}_p and T_p^*P as Lie subalgebras and it is isomorphic to $\mathfrak{g}_p \oplus T_p^*P$ as vector spaces. We can think of \mathfrak{l}_p as being built out of the Lie algebras \mathfrak{g}_p and T_p^*P together with the action of \mathfrak{g}_p on T_p^*P given by (40) and the action of T_p^*P on \mathfrak{g}_p by

$$T_p^*P \times \mathfrak{g}_p \longrightarrow \mathfrak{g}_p : (\alpha_p, x) \longmapsto ad_{\phi(\alpha_p)}^*x.$$

Example 4.11 Combining the situations in Example 4.9 and Example 4.10, we consider the case when P is a Poisson homogeneous G space and the Poisson bivector field π on P vanishes at some point $p \in P$. In this case, we can identify P with the quotient space G/G_p , where G_p is the stabilizer subgroup of G at p . The Poisson structure π is the unique one on G/G_p such that the projection from G to G/G_p is a Poisson map. From Example 4.10 we see that

$$\mathfrak{g}_p^\perp = \{\xi \in \mathfrak{g}^* : (\xi, x) = 0 \ \forall x \in \mathfrak{g}_p\} \subset \mathfrak{g}^*$$

is a Lie subalgebra of \mathfrak{g}^* . The Lie subalgebra \mathfrak{g}_p of \mathfrak{g} is said to be coisotropic relative to the Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ [Lu-We]. The Lie algebra \mathfrak{l}_p is now isomorphic to the Lie subalgebra $\mathfrak{g}_p + \mathfrak{g}_p^\perp$ of $\mathfrak{d} = \mathfrak{g} \ltimes \mathfrak{g}^*$.

The fact that the Lie algebra $\mathfrak{l}_p = \mathfrak{g}_p + \mathfrak{g}_p^\perp$ is a double Lie algebra will be used in Section 7 to calculate the relative Lie algebra cohomology of \mathfrak{l}_p relative to the Lie subalgebra \mathfrak{g}_p , which will be shown to be isomorphic to the G -invariant Poisson cohomology of the Poisson homogeneous space G/G_p . See Theorem 7.8.

A special case of this situation is when \mathfrak{g}_p^\perp is a Lie ideal of \mathfrak{g}^* . In such a case, G_p is a Poisson Lie subgroup of G . The Lie bracket on $\mathfrak{g}_p + \mathfrak{g}_p^\perp$ is a little simpler:

$$[x + \xi, y + \eta] = [x, y] + [\xi, \eta] + ad_x^*\eta - ad_y^*\xi \quad (42)$$

for $x, y \in \mathfrak{g}_p$ and $\xi, \eta \in \mathfrak{g}_p^\perp$. In particular, \mathfrak{g}_p^\perp is a Lie ideal of $\mathfrak{l}_p \cong \mathfrak{g}_p + \mathfrak{g}_p^\perp$.

This example will be carried out further in Example 7.6 in Section 7.

5 Examples

In this section, we give examples of the Lie algebroid $A = (P \times \mathfrak{g}) \ltimes T^*P$ constructed in Section 4.

Example 5.1 Any Lie group G can be regarded as a Poisson Lie group with the zero Poisson structure. In this case, an action of G on a Poisson manifold is Poisson if and only if each $g \in G$ acts on P as a Poisson diffeomorphism. In particular, each $x \in \mathfrak{g}$ acts on the space of 1-forms on P as a derivation. The corresponding Lie algebroid structure on the vector bundle $(P \times \mathfrak{g}) \oplus T^*P$ described in Theorem 4.1 becomes a semi-direct product Lie algebroid structure [H-M].

Example 5.2 Let G be a Poisson Lie group. Consider the left action of G on itself by left translations. This is a Poisson action by definition. By trivializing the cotangent bundle T^*G to $G \times \mathfrak{g}^*$ via,

$$T^*G \longrightarrow G \times \mathfrak{g}^* : \quad \xi_g \longmapsto \phi(\xi_g) = r_g^* \xi_g, \quad g \in G, \quad \xi_g \in T_g^*G,$$

we can trivialize the vector bundle $(P \times \mathfrak{g}) \oplus T^*G$ to $G \times (\mathfrak{g} \oplus \mathfrak{g}^*) = G \times \mathfrak{d}$ by the map

$$\Phi : (x, \xi_g) \longmapsto x + \phi(\xi_g) = x + r_g^* \xi_g, \quad g \in G, \quad \xi_g \in T_g^*G.$$

In this case, the Lie algebroid structure on the bundle A described in Theorem 4.1 is the transformation Lie algebroid structure on $G \times \mathfrak{d}$ given by the left infinitesimal action of \mathfrak{d} on G given by (15).

The maximal isotropic Lie subalgebra \mathfrak{l}_g of \mathfrak{d} in this case is simply the Lie subalgebra $Ad_g \mathfrak{g}^*$, the image of \mathfrak{g}^* under the action of g on \mathfrak{d} given by (18).

Example 5.3 Assume that the action $\sigma : G \times P \rightarrow P$ is transitive, so P is a Poisson homogeneous G -space. In this case, the map

$$T^*P \longrightarrow P \times \mathfrak{g}^* : \quad \alpha_p \longmapsto (p, \phi(\alpha_p))$$

embeds T^*P into $P \times \mathfrak{g}^*$ as a subbundle whose fiber over the point $p \in P$ is

$$\mathfrak{g}_p^\perp = \{ \xi \in \mathfrak{g}^* : (\xi, x) = 0, \quad \forall x \in \mathfrak{g}_p \}.$$

Thus via the map

$$\Phi : A = (P \times \mathfrak{g}) \bowtie T^*P \longrightarrow \mathfrak{d} = \mathfrak{g} \bowtie \mathfrak{g}^* : \quad (x, \alpha_p) \longmapsto x + \phi(\alpha_p)$$

we can also embed A into $P \times (\mathfrak{g} \oplus \mathfrak{g}^*)$ as a subbundle whose fiber over $p \in P$ is $\mathfrak{g} \oplus \mathfrak{g}_p^\perp$. Under such an embedding, a section of A has the form $a = \bar{x} + \bar{\xi}$ where $\bar{x} \in C^\infty(P, \mathfrak{g})$ and $\bar{\xi} \in C^\infty(P, \mathfrak{g}^*)$ is such that $\bar{\xi}(p) \in \mathfrak{g}_p^\perp$ for all $p \in P$. Denote by $\pi^\#(\bar{\xi})$ the vector field $\pi^\#(\alpha)$ if $\phi(\alpha) = \bar{\xi}$. The anchor of A is then the vector field $-\sigma_{\bar{x}} - \pi^\#(\bar{\xi})$. Since Φ is a Lie algebroid morphism from A to the Lie algebra $\mathfrak{d} = \mathfrak{g} \bowtie \mathfrak{g}^*$ by Proposition 4.4, the Lie bracket on the sections of A now looks like that for a transformation Lie algebroid:

$$\{\bar{x} + \bar{\xi}, \bar{y} + \bar{\eta}\} = [\bar{x} + \bar{\xi}, \bar{y} + \bar{\eta}]_{\mathfrak{d}} - (\sigma_{\bar{x}} + \pi^\#(\bar{\xi})) \cdot (\bar{y} + \bar{\eta}) + (\sigma_{\bar{y}} + \pi^\#(\bar{\eta})) \cdot (\bar{x} + \bar{\xi}). \quad (43)$$

Example 5.4 Consider now the example of the left dressing action of G on its dual Poisson Lie group (G^*, π_{G^*}) (see Example 4.8).

Identify T^*G^* with the trivial vector bundle $G^* \times \mathfrak{g}$ by left translations on G^* . Then the cotangent bundle Lie algebroid T^*G^* becomes the same as the transformation Lie algebroid $G^* \times \mathfrak{g}$ defined by the left dressing action of \mathfrak{g} on G^* . Thus the Lie algebroid A , whose underlying vector bundle is now identified with the trivial vector bundle $G^* \times (\mathfrak{g} \oplus \mathfrak{g})$, is built out of two copies of the transformation Lie algebroid $G^* \times \mathfrak{g}$ and a pair of representations D and D' of it on itself. These two representations are respectively given by

$$D_{\bar{x}}\bar{y} = -\sigma_{\bar{x}} \cdot \bar{y} + [\bar{x}, \bar{y}]_{\mathfrak{g}} + ad_{\tau(\bar{y})}^* \bar{x} \quad (44)$$

$$D'_{\bar{y}}\bar{x} = -\sigma_{\bar{y}} \cdot \bar{x} + ad_{\tau(\bar{y})}^* \bar{x}, \quad (45)$$

where

$$\tau : C^\infty(G^*, \mathfrak{g}) \longrightarrow C^\infty(G^*, \mathfrak{g}^*) : \tau(\bar{y})(u) = \bar{y}(u) \lrcorner (l_{u^{-1}} \pi_{G^*}(u)), \quad u \in G^*. \quad (46)$$

The resulting Lie algeroid structure on $G^* \times (\mathfrak{g} \oplus \mathfrak{g})$ has the property that the bundle map

$$G^* \times (\mathfrak{g} \oplus \mathfrak{g}) \longrightarrow G^* \times \mathfrak{g} : (x_u, y_u) \longmapsto x_u + y_u$$

is a Lie algebroid homomorphism.

6 Proof of Theorem

We give the proof of Theorem 4.1 in this section. Following the definition of a Lie algebroid, we need to prove

1) for any $f \in C^\infty(P)$, $\bar{x} \in C^\infty(P, \mathfrak{g})$ and any 1-form α on P ,

$$\begin{aligned}\{f\bar{x}, \alpha\} &= f\{\bar{x}, \alpha\} + (\pi^\#(\alpha)f)\bar{x} \\ \{\bar{x}, f\alpha\} &= f\{\bar{x}, \alpha\} - (\sigma_{\bar{x}}f)\alpha;\end{aligned}$$

2) the bracket on the sections of A described in Theorem 4.1 satisfies the Jacobi identity;

3) the map $-\sigma - \pi^\#$ is a Lie algebra homomorphism from the Lie algebra of sections of A to the Lie algebra of vector fields on P .

1) follows from the fact that $D_\alpha \bar{x}$ and $D_{\bar{x}} \alpha$ define representations of Lie algebroids. This fact can also be used to reduce 2) to Lemma 6.1 and 3) to Lemma 6.2 as follows.

Lemma 6.1 *For any $\bar{x}, \bar{y} \in C^\infty(P, \mathfrak{g})$ and any 1-forms α and β on P , we have*

$$D_\alpha \{\bar{x}, \bar{y}\} = \{D_\alpha \bar{x}, \bar{y}\} + \{\bar{x}, D_\alpha \bar{y}\} + D_{D_{\bar{y}} \alpha} \bar{x} - D_{D_{\bar{x}} \alpha} \bar{y} \quad (47)$$

$$D_{\bar{x}} \{\alpha, \beta\} = \{D_{\bar{x}} \alpha, \beta\} + \{\alpha, D_{\bar{x}} \beta\} + D_{D_\beta \bar{x}} \alpha - D_{D_\alpha \bar{x}} \beta. \quad (48)$$

Lemma 6.2 *For any $\bar{x} \in C^\infty(P, \mathfrak{g})$ and any 1-form α on P , we have*

$$\sigma_{D_\alpha \bar{x}} - \pi^\#(D_{\bar{x}} \alpha) = [\sigma_{\bar{x}}, \pi^\#(\alpha)]. \quad (49)$$

We prove Lemma 6.2 first, since it is easier to prove and it will be used in the proof of Lemma 6.1.

Proof of Lemma 6.2 Fix the 1-form α on P . For any $\bar{x} \in C^\infty(P, \mathfrak{g})$, set

$$V_\alpha(\bar{x}) = \sigma_{D_\alpha \bar{x}} - \pi^\#(D_{\bar{x}} \alpha) - [\sigma_{\bar{x}}, \pi^\#(\alpha)].$$

It follows from the axioms for Lie algebroid representations that

$$V_\alpha(f\bar{x}) = fV_\alpha(\bar{x})$$

for any $f \in C^\infty(P)$. Thus it suffices to show that $V_\alpha(x) = 0$ for any constant function $x \in C^\infty(P, \mathfrak{g})$ corresponding to $x \in \mathfrak{g}$. In other words, we need to show that

$$\sigma_{ad_{\phi(\alpha)}^* x} + \pi^\#(L_{\sigma_x} \alpha) - [\sigma_x, \pi^\#(\alpha)] = 0 \quad (50)$$

for any $x \in \mathfrak{g}$. By pairing the left hand side with an arbitrary 1-form β on P , we see that this is equivalent to

$$(L_{\sigma_x} \pi)(\alpha, \beta) = (x, [\phi(\alpha), \phi(\beta)]_{\mathfrak{g}^*}),$$

which is just the infinitesimal condition (21) for the action σ to be Poisson.

Q.E.D.

Proof of Lemma 6.1. For \bar{x} , \bar{y} , α and β as given, set

$$\begin{aligned} V_\alpha(\bar{x}, \bar{y}) &= D_\alpha\{\bar{x}, \bar{y}\} - \{D_\alpha\bar{x}, \bar{y}\} - \{\bar{x}, D_\alpha\bar{y}\} - D_{D_{\bar{y}}\alpha}\bar{x} + D_{D_{\bar{x}}\alpha}\bar{y} \\ W_{\alpha,\beta}(\bar{x}) &= D_{\bar{x}}\{\alpha, \beta\} - \{D_{\bar{x}}\alpha, \beta\} - \{\alpha, D_{\bar{x}}\beta\} - D_{D_\beta\bar{x}}\alpha + D_{D_\alpha\bar{x}}\beta. \end{aligned}$$

Using Identity (50) and the axioms for Lie algebroid representations, we see that

$$V_\alpha(f\bar{x}, \bar{y}) = V_\alpha(\bar{x}, f\bar{y}) = fV_\alpha(\bar{x}, \bar{y})$$

for any $f \in C^\infty(P)$. Thus it suffices to show that $V_\alpha(x, y) = 0$ for any constant functions $x, y \in C^\infty(P, \mathfrak{g})$ corresponding to $x, y \in \mathfrak{g}$. But this can be deduced from the following fact about the Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$:

$$ad_\xi^*[x, y] = [ad_\xi^*x, y] + [x, ad_\xi^*y] + ad_{ad_y^*\xi}x - ad_{ad_x^*\xi}y \quad (51)$$

for any $x, y \in \mathfrak{g}$ and $\xi \in \mathfrak{g}^*$. The proof of (51) can be obtained from the explicit formula (12) for the Lie bracket on the double \mathfrak{d} of $(\mathfrak{g}, \mathfrak{g}^*)$. It can also be found in [Am].

For $W_{\alpha,\beta}$, it again follows from the axioms for Lie algebroid representations that

$$W_{\alpha,\beta}(f\bar{x}) = fW_{\alpha,\beta}(\bar{x})$$

for any $f \in C^\infty(P)$. Thus it suffices to show that $W_{\alpha,\beta}(x) = 0$ for constant functions $x \in C^\infty(P, \mathfrak{g})$ corresponding to $x \in \mathfrak{g}$. In other words, we need to show that

$$L_{\sigma_x}\{\alpha, \beta\} - \{L_{\sigma_x}\alpha, \beta\} - \{\alpha, L_{\sigma_x}\beta\} + D_{\tilde{x}_\beta}\alpha - D_{\tilde{x}_\alpha}\beta = 0,$$

where

$$\tilde{x}_\beta = ad_{\phi(\beta)}^*x, \quad \tilde{x}_\alpha = ad_{\phi(\alpha)}^*x.$$

Using Formula (3) for the Lie bracket on 1-forms on P and Identity (50), we can reduce $W_{\alpha,\beta}(x) = 0$ to

$$(L_{\sigma_x}\pi)(\alpha, \beta) = (x, [\phi(\alpha), \phi(\beta)]_{\mathfrak{g}^*})$$

which is nothing but the infinitesimal criterion (21) for σ being Poisson.

Q.E.D.

7 Poisson homogeneous spaces

In this section, we assume that G is a connected Poisson Lie group and that P is a homogeneous G -space with a Poisson structure π such that the action of G on P is Poisson. We will treat two aspects of such a Poisson manifold: its symplectic leaves and its G -invariant Poisson cohomology. We first look at its symplectic leaves.

For each $p \in P$, embed \mathfrak{l}_p into \mathfrak{d} via

$$\Phi : \mathfrak{l}_p \longrightarrow \mathfrak{d} : (x, \alpha_p) \longmapsto x + \phi(\alpha).$$

Then we get a Lie algebra homomorphism from \mathfrak{l}_p to $\chi(G)$:

$$\mathfrak{l}_p \longrightarrow \chi(G) : (x, \alpha_p) \longmapsto \lambda_{x+\phi(\alpha_p)}(g) = r_g \left(Ad_g x + (Ad_{g^{-1}}^* \phi(\alpha_p) \lrcorner (r_{g^{-1}} \pi_G(g))) \right).$$

Recall (see (17) in Section 3) that the vector fields $\lambda_{x+\xi}$, for $x \in \mathfrak{g}$, $\xi \in \mathfrak{g}^*$, define a right action of \mathfrak{d} on G . Set

$$\sigma_p : G \longrightarrow P : g \longmapsto gp. \tag{52}$$

Lemma 7.1 *For any $p \in P$, $g \in G$, the image of the subspace*

$$\{\lambda_{x+\phi(\alpha_p)}(g) : (x, \alpha_p) \in \mathfrak{l}_p\} \subset T_g G$$

of $T_g G$ under the differential of the map σ_p is exactly the subspace $im(\pi^\#|_{T_{gp}^ P})$ of $T_{gp} P$.*

Proof. We know from Proposition 4.3 and Proposition 4.5 that

$$\sigma_p(\lambda_{x+\phi(\alpha_p)}(g)) = -\pi^\#((g^{-1})^* \alpha_p) \in im(\pi^\#|_{T_{gp}^* P}).$$

The fact that we get all elements in $im(\pi^\#|_{T_{gp}^* P})$ this way is because the action of G on P is transitive.

Q.E.D.

We now have the following description of the symplectic leaves in P .

Theorem 7.2 *Fix any $p \in P$. Assume that L_p is a connected Lie group and that the vector fields*

$$\lambda_{x+\phi(\alpha_p)}, \quad (x, \alpha_p) \in \mathfrak{l}_p$$

integrate to a right action of L_p on G . Then the images of the L_p orbits in G under the map

$$\sigma_p : G \longrightarrow P : g \longmapsto gp$$

give all the symplectic leaves in P .

Proof. By definition, the symplectic leaves of P are the integral submanifolds of the distribution $\{im(\pi^\#|_{T_p^*P})\}_{p \in P}$ on P . By Lemma 7.1, we know that this distribution is the image under σ_p of the distribution on G defined by the vector fields $\lambda_{x+\phi(\alpha_p)}, (x, \alpha_p) \in \mathfrak{l}_p$, on G . The integral submanifolds of the latter distribution are exactly all the L_p -orbits on G . Thus these orbits project to P under σ_p to give all the symplectic leaves in P .

Q.E.D.

Remark 7.3 Note that when the double group D of G and G^* have a global decomposition $D = GG^*$, the right action of L_p on G described in Theorem 7.2 is simply the restriction to L_p of the right action of D on $G = G^* \backslash D$ (see (16) in Section 3).

We now turn to the G -invariant Poisson cohomology of P . Recall that the Poisson cohomology of P is by definition the Lie algebroid cohomology of the cotangent bundle Lie algebroid T^*P with trivial coefficients. The corresponding coboundary operator on $\wedge^\bullet TP$ is also given by

$$V \longmapsto [\pi, V]$$

where $[\ , \]$ denotes the Schouten bracket [Ku]. If V is G -invariant, then so is $[\pi, V]$. This is because

$$L_{\sigma_x}[\pi, V] = [L_{\sigma_x}, V] = [\sigma_{\delta(x)}, V] = 0$$

for any $x \in \mathfrak{g}$ (see (21)). Hence, since G is connected, the space $\Gamma(\wedge TP)^G$ of G -invariant multi-vector fields on P is closed under the coboundary operator $[\pi, \bullet]$.

Definition 7.4 The cohomology of $(\Gamma(\wedge TP)^G, [\pi, \bullet])$ is called the G -invariant Poisson cohomology of P . We denote it by $H_{\pi, G}(P)$.

Assume now that P is a Poisson homogeneous G -space. We have seen in Example 4.9 that the kernel of the anchor map $-\sigma - \pi^\#$ at each $p \in P$ is a Lie algebra \mathfrak{l}_p of dimension

$n = \dim \mathfrak{g}$. Let G_p be the stabilizer Lie subgroup of G at p , and let \mathfrak{g}_p be its Lie algebra. The group G_p acts on \mathfrak{l}_p via the action of G on A given by (28), and the corresponding action of \mathfrak{g}_p is the adjoint action of \mathfrak{g}_p on \mathfrak{l}_p via the Lie algebra embedding $\mathfrak{g}_p \subset \mathfrak{l}_p$. Denote by $H^\bullet(\mathfrak{l}_p, G_p)$ the relative Lie algebra cohomology of \mathfrak{l}_p relative to G_p .

Theorem 7.5

$$H_{\pi, G}^\bullet(P) \cong H^\bullet(\mathfrak{l}_p, G_p)$$

for every $p \in P$.

Proof. Denote by L the kernel of $-\sigma - \pi$, so L is a subbundle of A whose fiber at $p \in P$ is the Lie algebra \mathfrak{l}_p . By taking fiber-wise Lie bracket of sections of L , we can think of L as a Lie algebroid itself with the zero anchor map. As a such, it is a “totally intransitive” Lie algebroid [Mc21]. The inclusion of L into A makes it into a Lie subalgebroid of A . We know from Theorem 4.1 and Proposition 4.5 that L is a G -vector bundle and that the action of G on L preserves the Lie brackets on the fibers of L .

Consider now the coboundary operator d_L for the Lie algebroid cohomology of L with trivial coefficients (see (8)). Since L has the zero anchor map, the operator d_L is $C^\infty(P)$ -linear, i.e., d_L is given by the field $d_{\mathfrak{l}_p}, p \in P$, of fiber-wise operators, where $d_{\mathfrak{l}_p}$ is the coboundary operator for the cohomology of the Lie algebra \mathfrak{l}_p .

For each $p \in P$, set

$$\mathfrak{g}_p^0 = \{f \in \mathfrak{l}_p^* : f(x) = 0, \forall x \in \mathfrak{g}_p\} \subset \mathfrak{l}_p^*. \quad (53)$$

Then the subspace $(\wedge^\bullet \mathfrak{g}_p^0)^{G_p}$ of G_p -invariant vectors in $\wedge^\bullet \mathfrak{g}_p^0$ is invariant under $d_{\mathfrak{l}_p}$. The relative Lie algebra cohomology of \mathfrak{l}_p relative to G_p is by definition the cohomology of the cochain complex $((\wedge^\bullet \mathfrak{g}_p^0)^{G_p}, d_{\mathfrak{l}_p})$.

Denote by L_σ^0 the subbundle of L^* whose fiber at p is \mathfrak{g}_p^0 . It is a G -invariant subbundle of L^* . By dimension counting, the map

$$TP \longrightarrow L^* : v_p \longmapsto f_{v_p} : (x, \alpha_p) \longmapsto (v_p, \alpha_p)$$

gives a bundle isomorphism from TP to L_σ^0 . It is also a G -bundle morphism. Therefore we get a vector space isomorphism

$$\Gamma(\wedge^k TP)^G \longrightarrow \Gamma(\wedge^k L_\sigma^0)^G \subset \Gamma(\wedge^k L^*), \quad (54)$$

where both sides denote G -invariant sections in the corresponding vector bundles. Denote this map by θ . It remains to show that

$$d_L \theta(V) = \theta([\pi, V])$$

for any $V \in \Gamma(\wedge^k TP)^G$, for then θ would give a cochain complex isomorphism from $\Gamma(\wedge^k TP)^G$ to $((\wedge^\bullet \mathfrak{g}_p^0)^{G_p}, d_{\iota_p})$ for each $p \in P$, and thus inducing an isomorphism between their cohomology spaces.

Let V be a G -invariant k -vector field on P . Then

$$D_{\bar{x}} V = 0, \quad \forall \bar{x} \in C^\infty(P, \mathfrak{g}).$$

Consequently, for any 1-forms $\alpha_i, i = 1, \dots, k$, on P , we have

$$-\sigma_{\bar{x}}(V, \alpha_1 \wedge \dots \wedge \alpha_k) = (V, D_{\bar{x}}(\alpha_1 \wedge \dots \wedge \alpha_k)).$$

It follows then that for any $l_i = (\bar{x}_i, \alpha_i) \in \Gamma(L), i = 1, \dots, k$

$$\begin{aligned} (d_L \theta(V))(l_1 \wedge \dots \wedge l_k) &= \sum_{i < j} (-1)^{i+j} (\theta(V), \{l_i, l_j\} \wedge l_1 \wedge \dots \wedge \hat{l}_i \dots \wedge \hat{l}_j \dots \wedge l_k) \\ &= \sum_{i < j} (-1)^{i+j} (V, \{\alpha_i, \alpha_j\} \wedge \alpha_1 \wedge \dots \wedge \hat{\alpha}_i \dots \wedge \hat{\alpha}_j \dots \wedge \alpha_k) \\ &\quad + \sum_{i < j} (-1)^{i+j} (V, (D_{\bar{x}_i} \alpha_j - D_{\bar{x}_j} \alpha_i) \wedge \alpha_1 \wedge \dots \wedge \hat{\alpha}_i \dots \wedge \hat{\alpha}_j \dots \wedge \alpha_k) \\ &= \sum_{i < j} (-1)^{i+j} (V, \{\alpha_i, \alpha_j\} \wedge \alpha_1 \wedge \dots \wedge \hat{\alpha}_i \dots \wedge \hat{\alpha}_j \dots \wedge \alpha_k) \\ &\quad + \sum_j (-1)^j (V, D_{\bar{x}_j}(\alpha_1 \wedge \dots \wedge \hat{\alpha}_j \dots \wedge \alpha_k)) \\ &= \sum_{i < j} (-1)^{i+j} (V, \{\alpha_i, \alpha_j\} \wedge \alpha_1 \wedge \dots \wedge \hat{\alpha}_i \dots \wedge \hat{\alpha}_j \dots \wedge \alpha_k) \\ &\quad + \sum_j (-1)^{j+1} \sigma_{\bar{x}_j}(V, \alpha_1 \wedge \dots \wedge \hat{\alpha}_j \dots \wedge \alpha_k) \\ &= \sum_{i < j} (-1)^{i+j} (V, \{\alpha_i, \alpha_j\} \wedge \alpha_1 \wedge \dots \wedge \hat{\alpha}_i \dots \wedge \hat{\alpha}_j \dots \wedge \alpha_k) \\ &\quad + \sum_j (-1)^{j+1} (-\pi^\#(\alpha_j))(V, \alpha_1 \wedge \dots \wedge \hat{\alpha}_j \dots \wedge \alpha_k) \\ &= [\pi, V](\alpha_1 \wedge \dots \wedge \alpha_k) \\ &= \theta([\pi, V])(l_1 \wedge \dots \wedge l_k). \end{aligned}$$

Therefore

$$d_L \theta(V) = \theta([\pi, V]).$$

This finishes the proof of the theorem.

Q.E.D.

Example 7.6 We continue with Example 4.11. Let G be a connected Poisson Lie group, and let $H \subset G$ (denoted by G_p in Example 4.11) be a connected closed subgroup of G with Lie algebra \mathfrak{h} such that

$$\mathfrak{h}^\perp = \{\xi \in \mathfrak{g}^* : (\xi, x) = 0, \forall x \in \mathfrak{h}\} \subset \mathfrak{g}^*$$

is a Lie subalgebra of \mathfrak{g}^* . In this case, there is a unique Poisson structure on the quotient space G/H such that the projection from G to G/H is a Poisson map, and the left action of G on G/H by left translations is a Poisson action. We have seen in Example 4.11 that the Lie algebra \mathfrak{l}_p corresponding to the point $p = eH$ in G/H is the Lie subalgebra $\mathfrak{h} + \mathfrak{h}^\perp$ of $\mathfrak{d} = \mathfrak{g} \ltimes \mathfrak{g}^*$. By Theorem 7.5, we know that the G -invariant Poisson cohomology of G/H is isomorphic to the relative Lie algebra cohomology of $\mathfrak{h} + \mathfrak{h}^\perp$ relative to the Lie subalgebra \mathfrak{h} . We also observed in Example 4.11 that the Lie algebra $\mathfrak{h} + \mathfrak{h}^\perp$ is an example of a double Lie algebra. We first prove the following fact on relative cohomology in the case of a double Lie algebra.

Lemma 7.7 *Let \mathfrak{l} be a Lie algebra and let \mathfrak{h} and \mathfrak{n} be Lie subalgebras of \mathfrak{l} such that*

$$\mathfrak{l} = \mathfrak{h} \oplus \mathfrak{n}$$

as vector spaces. For $x \in \mathfrak{h}$ and $\xi \in \mathfrak{n}$, write

$$[x, \xi] = -\xi \cdot x + x \cdot \xi$$

where $\xi \cdot x \in \mathfrak{h}$ and $x \cdot \xi \in \mathfrak{n}$. The maps

$$\mathfrak{h} \times \mathfrak{n} \longrightarrow \mathfrak{n} : (x, \xi) \longmapsto x \cdot \xi \tag{55}$$

$$\mathfrak{n} \times \mathfrak{h} \longrightarrow \mathfrak{h} : (\xi, x) \longmapsto \xi \cdot x \tag{56}$$

define a pair of actions of \mathfrak{h} and \mathfrak{n} on each other. Denote by $(\wedge \mathfrak{n}^*)^{\mathfrak{h}}$ the subspace of $\wedge \mathfrak{n}^*$ of \mathfrak{h} -invariant vectors with respect to the action of \mathfrak{h} on \mathfrak{n}^* contragradient to the action of \mathfrak{h} on \mathfrak{n} . Let

$$d_{\mathfrak{n}} : \wedge^k \mathfrak{n}^* \longrightarrow \wedge^{k+1} \mathfrak{n}^*$$

be the Chevalley-Eilenberg coboundary operator for the Lie algebra \mathfrak{n} . Then the subspace $(\wedge \mathfrak{n}^*)^{\mathfrak{h}}$ of $\wedge \mathfrak{n}^*$ is invariant under $d_{\mathfrak{n}}$ (even though the action of \mathfrak{h} on \mathfrak{n} is not by Lie algebra derivations). The cohomology $H^\bullet((\wedge \mathfrak{n}^*)^{\mathfrak{h}}, d_{\mathfrak{n}})$ of the cochain subcomplex $((\wedge \mathfrak{n}^*)^{\mathfrak{h}}, d_{\mathfrak{n}})$ is isomorphic to the relative Lie algebra cohomology of \mathfrak{l} relative to the Lie subalgebra \mathfrak{h} .

Proof. By embedding \mathfrak{n}^* into \mathfrak{l}^* via

$$\mathfrak{n}^* \ni f \longmapsto \bar{f}(x + y) = f(y), \quad x \in \mathfrak{h}, y \in \mathfrak{n}$$

we can identify \mathfrak{n}^* with $\mathfrak{h}^\perp \subset \mathfrak{l}^*$ and $(\wedge \mathfrak{n}^*)^{\mathfrak{h}}$ with $(\wedge \mathfrak{h}^\perp)^{\mathfrak{h}} \subset \wedge \mathfrak{l}^*$. Denote again by f an arbitrary element in $(\wedge \mathfrak{n}^*)^{\mathfrak{h}}$ and by \bar{f} its image in $(\wedge \mathfrak{h}^\perp)^{\mathfrak{h}} \subset \wedge \mathfrak{l}^*$, i.e.,

$$\bar{f}(x_1 + y_1, \dots, x_k + y_k) = f(y_1, \dots, y_k).$$

We need to show that

$$d_{\mathfrak{l}} \bar{f} = \overline{d_{\mathfrak{n}} f},$$

where $d_{\mathfrak{l}} : \text{wedge}^k \mathfrak{l}^* \rightarrow \wedge^{k+1} \mathfrak{l}^*$ is the Chevalley-Eilenberg coboundary operator for \mathfrak{l} . Let $l_i = x_i + y_i \in \mathfrak{h} + \mathfrak{n}$ with $x_i \in \mathfrak{h}$ and $y_i \in \mathfrak{n}$, $i = 1, \dots, k+1$. Then

$$\begin{aligned} (d_{\mathfrak{l}} \bar{f})(l_1, \dots, l_{k+1}) &= \sum_{i < j} (-1)^{i+j} \bar{f}([l_i, l_j], \dots, \hat{l}_i, \dots, \hat{l}_j, \dots, l_{k+1}) \\ &= \sum_{i < j} (-1)^{i+j} f([y_i, y_j] + x_i \cdot y_j - x_j \cdot y_i, \dots, \hat{y}_i, \dots, \hat{y}_j, \dots, y_{k+1}) \\ &= \overline{d_{\mathfrak{n}} f}(l_1, \dots, l_{k+1}) + \sum_j (-1)^j f(x_j \cdot (y_1 \wedge \dots \wedge \hat{y}_j \wedge \dots \wedge y_{k+1})) \\ &= \overline{d_{\mathfrak{n}} f}(l_1, \dots, l_{k+1}). \end{aligned}$$

In the last step we use that fact that f is \mathfrak{h} -invariant.

Q.E.D.

The following Theorem now follows from Lemma 7.7.

Theorem 7.8 *Let G be a connected Poisson Lie group, and let $H \subset G$ be a connected closed subgroup of G with Lie algebra \mathfrak{h} such that*

$$\mathfrak{h}^\perp = \{\xi \in \mathfrak{g}^* : (\xi, x) = 0, \forall x \in \mathfrak{h}\} \subset \mathfrak{g}^*$$

is a Lie subalgebra of \mathfrak{g}^ . Denote by $(\wedge(\mathfrak{h}^\perp)^*)^\mathfrak{h}$ the subspace in $\wedge(\mathfrak{h}^\perp)^*$ of \mathfrak{h} -invariant vectors with respect to the adjoint action of \mathfrak{h} on $(\mathfrak{h}^\perp)^* \cong \mathfrak{g}/\mathfrak{h}$. Let*

$$d_{\mathfrak{h}^\perp} : \wedge^k(\mathfrak{h}^\perp)^* \longrightarrow \wedge^{k+1}(\mathfrak{h}^\perp)^*$$

be the Chevalley-Eilenberg coboundary operator for the Lie algebra \mathfrak{h}^\perp . Then $(\wedge(\mathfrak{h}^\perp)^)^\mathfrak{h}$ is invariant under $d_{\mathfrak{h}^\perp}$, and the cohomology of $((\wedge(\mathfrak{h}^\perp)^*)^\mathfrak{h}, d_{\mathfrak{h}^\perp})$ is isomorphic to the G -invariant Poisson cohomology of the Poisson homogeneous space G/H .*

We now apply Theorem 7.8 to the Bruhat Poisson structure on a generalized flag manifold [Lu-We].

Let G be a connected finite dimensional semisimple complex Lie group and let $G = KAN$ be an Iwasawa decomposition of G as a real semisimple Lie group. Then there are natural Poisson structures on K and on AN making them into Poisson Lie groups. Moreover, as Poisson Lie groups, they are dual to each other, and the group G is the double of K and AN . On the Lie algebra level, we have the Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$. The imaginary part of the (complex-valued) Killing form of \mathfrak{g} gives a nondegenerate pairing between \mathfrak{k} and $\mathfrak{a} + \mathfrak{n}$. In other words, the pair $(\mathfrak{k}, \mathfrak{a} + \mathfrak{n})$ form a Lie bialgebra and \mathfrak{g} is its double Lie algebra.

Let T be the maximal torus in K , and let \mathfrak{t} be its Lie algebra. The subspace

$$\mathfrak{t}^\perp = \{\xi \in \mathfrak{k}^* : (\xi, x) = 0, \forall x \in \mathfrak{t}\}$$

of $\mathfrak{k}^* \cong \mathfrak{a} + \mathfrak{n}$ is clearly \mathfrak{n} which is an ideal in $\mathfrak{a} + \mathfrak{n}$. Thus there is a unique Poisson structure on K/T such that the projection from K to K/T is a Poisson map. Moreover, the left action of K on K/T is a Poisson action, so K/T is a Poisson homogeneous K -space.

Symplectic leaves in K/T for this Poisson structure are known to be exactly the Bruhat cells in K/T (thus the name Bruhat Poisson structure). This can be seen directly from Theorem 7.2, for in this case, the Lie group L_p , for $p = eT \in K/T$, is TN , and the right action of it on K described in Theorem 7.2 is just the restriction to TN of the right action of G on $K \cong (AN) \backslash G$ by right translations. The projection of these orbits from K to K/T are exactly the Bruhat cells in K/T .

Consider now the K -invariant Poisson cohomology space of the Bruhat Poisson structure with complex coefficients. By Theorem 7.8, we get the following

Theorem 7.9 *The (complex-valued) K -invariant Poisson cohomology $H_{\pi,K}(K/T)$ of the Bruhat Poisson structure on K/T is isomorphic to the space $End_{\mathfrak{h}}(H(\mathfrak{n}))$, where \mathfrak{h} is the complexification of \mathfrak{t} , and \mathfrak{n} here is considered as a complex Lie algebra. Thus the dimension of $H_{\pi,K}^k(K/T)$ is 0 if k is odd and is equal to the number of Weyl group element of length $k/2$ if k is even. In particular, it is isomorphic to the (complex-valued) de Rham cohomology of K/T .*

Proof. From Theorem 7.8 we know that

$$H_{\pi,K}(K/T) \cong H((\wedge(\mathfrak{n} + \mathfrak{n}_-)^*)^{\mathfrak{h}}, d_{\mathfrak{n}}) \cong End_{\mathfrak{h}}(H(\mathfrak{n})).$$

By Kostant's theorem [Kt1], there is exactly one cohomology class for the Lie algebra \mathfrak{n} corresponding to each element in the Weyl group W of K . Choose root vectors $\{E_{\alpha}, E_{-\alpha} : \alpha > 0\}$. For each $w \in W$, set

$$\Phi_w := \{\beta > 0 : w^{-1}\beta > 0\} = \{\beta_1, \dots, \beta_k\}.$$

Then the (complex) K -invariant $2k$ -vector field

$$E_{-\beta_1} \wedge \dots \wedge E_{-\beta_k} \wedge E_{\beta_1} \wedge \dots \wedge E_{\beta_k}$$

on K/T is a representative of a cohomology class of the K -invariant Poisson cohomology of the Bruhat Poisson structure on K/T . The fact that the space $End_{\mathfrak{h}}(H(\mathfrak{n}))$ is isomorphic to the de Rham cohomology of K/T is another theorem of Kostant's [Kt2].

8 The Lie groupoid of A

We first recall the definition of a Lie groupoid. Details can be found in [Mc1].

A groupoid over a set P is a set M , together with

- (1) surjections $s, t : M \rightarrow P$ (called the source and target maps respectively);
- (2) $\mu : M * M \rightarrow M$ (called the multiplication), where

$$M * M = \{(m_2, m_1) \in M \times M : s(m_2) = t(m_1)\};$$

each pair (m_2, m_1) in $M * M$ is said to be “composable”;

(3) an injection $\varepsilon : P \rightarrow M$ (identities);

(4) $\iota : M \rightarrow M$ (inversion).

These maps must satisfy

(1) associative law: $\mu(\mu(m_3, m_2), m_1) = \mu(m_3, \mu(m_2, m_1))$ (if one side is defined, so is the other);

(2) identities: for each $m \in M$, $(\varepsilon(t(m)), m) \in M * M$, $(m, \varepsilon(s(m))) \in M * M$, and $\mu(\varepsilon(t(m)), m) = \mu(m, \varepsilon(s(m))) = m$;

(3) inverses: for each $m \in M$, $(m, \iota(m)) \in M * M$, $(\iota(m), m) \in M * M$, and $\mu(m, \iota(m)) = \varepsilon(t(m))$ and $\mu(\iota(m), m) = \varepsilon(s(m))$.

A Lie groupoid (or differential groupoid) M over a manifold P is a groupoid with a differential structure such that (1) s and t are differentiable submersions (this implies that $M * M$ is a submanifold of $M \times M$), and (2) μ, ε and ι are differentiable maps.

Given a Lie groupoid M over P , the normal bundle of P in M has a Lie algebroid structure over P whose sections can be identified with left invariant vector fields on M . It is called the Lie algebroid of M .

A left action of a Lie groupoid (M, P, s, t) on a smooth submersion $f : Q \rightarrow P$ is a map

$$M *_f Q = \{(m, q) : s(m) = f(q)\} \longrightarrow Q : (m, q) \longmapsto mq$$

such that

$$(1) f(mq) = t(m)$$

$$(2) m_2(m_1q) = (m_2m_1)q$$

$$(3) \varepsilon(f(q))q = q$$

for all $m, m_1, m_2 \in M$ and $q \in Q$ which are suitably compatible.

Given such an action, one can construct the transformation groupoid on $M *_f Q$ with base Q by defining

$$\begin{aligned} s'(m, q) &= q, & t'(m, q) &= mq \\ (m_2, m_1q)(m_1, q) &= (m_2m_1, q). \end{aligned}$$

The map

$$M *_f Q \rightarrow M : (m, q) \longmapsto m$$

is a morphism of Lie groupoid over $f : Q \rightarrow P$.

A right action of a groupoid and the corresponding transformation groupoid are similarly defined.

Example 8.1 A Lie group is a Lie groupoid over a one point space. A Lie group action $G \times P \rightarrow P$ is an example of a Lie groupoid action. The Lie algebroid of the corresponding transformation groupoid is the transformation Lie algebroid discussed in Section 2.

Example 8.2 Let P be a Poisson manifold. We have seen in Section 2 that the cotangent bundle T^*P of P has a Lie algebroid structure. If there is a Lie groupoid (N, s_N, t_N) over P whose Lie algebroid is T^*P and whose s_N -fibers are simply-connected, we say that P is integrable as a Poisson manifold. It turns out that there is always a symplectic structure on N making it into a symplectic groupoid [C-D-W]. We call N the symplectic groupoid of P .

Assume now that G is a Poisson Lie group, P is an integrable Poisson manifold with symplectic groupoid (N, P, s_N, t_N) , and $\sigma : G \times P \rightarrow P$ is a Poisson action. Denote by (M, s_M, t_M) the transformation Lie groupoid over P defined by σ . The purpose of this section is to describe a Lie groupoid whose Lie algebroid is the Lie algebroid $A = (P \times \mathfrak{g}) \bowtie T^*P$ given in Theorem 4.1. Not surprisingly, this groupoid will be built out of M and N and a pair of actions of M and N on each other.

Recall that the map

$$\phi : T^*P \longrightarrow \mathfrak{g}^* : (\phi(\alpha_p), x) = (\alpha_p, \sigma_x(p))$$

is a Lie algebroid homomorphism. Assume that it can be integrated to a Lie groupoid homomorphism

$$\varphi : N \longrightarrow G^*,$$

where G^* is the dual Poisson Lie group of G (see [Xu]). The map φ is also a Poisson map. As a such, it induces a left action of G on N [Lu2] which we will denote by $(g, n) \longmapsto gn$. The map φ and the action of G on N have the following properties: for any $g \in G$, $n, n_1, n_2 \in N$,

$$\begin{aligned} t_N(gn) &= gt_N(n) \\ s_N(gn) &= g^{\varphi(n)} s_N(n) \end{aligned}$$

$$\begin{aligned}
\varepsilon_N(gn) &= g\varepsilon_N(n) \\
g(n_2n_1) &= (gn_2)(g^{\varphi(n_2)}n_1) \\
\varphi(\varepsilon_N(p)) &= 1 \in G^* \\
\varphi(n_2n_1) &= \varphi(n_2)\varphi(n_1) \\
\varphi(gn) &= {}^g\varphi(n).
\end{aligned}$$

Here gu and g^u , for $g \in G$ and $u \in G^*$, denote respectively the left action of G on G^* and the right action of G^* on G defined by the decomposition

$$gu = {}^gu g^u \in D$$

in the group D with Lie algebra \mathfrak{d} (see Section 3).

We can now describe the pair of actions of the groupoids M and N on each other.

The left action of M on $t_N : N \rightarrow P$ is given by

$$M *_{t_N} N \longrightarrow N : ((g, t_N(n)), n) \longmapsto gn. \quad (57)$$

The right action of N on M is given by

$$M *_{s_M} N = M *_{t_N} N : ((g, t_N(n)), n) \longmapsto (g^{\varphi(n)}, s_N(n)). \quad (58)$$

For notational simplicity, we denote the manifold $M *_{t_N} N = M *_{s_M} N$ by $M * N$ and denote the two actions respectively by

$$\begin{aligned}
M * N &\longrightarrow N : (m, n) \longmapsto {}^m n \\
M * N &\longrightarrow M : (m, n) \longmapsto m^n.
\end{aligned}$$

This pair of actions are compatible in the following sense:

- (1) $s_N({}^m n) = t_M(m^n), \quad \forall (m, n) \in M * N$
- (2) ${}^m(\varepsilon_N(s_M(m))) = \varepsilon_N(t_M(m)), \quad \forall m \in M$
- (3) $(\varepsilon_M(t_N(n)))^n = \varepsilon_M(s_N(n)), \quad \forall n \in N$
- (4) ${}^m(n_2n_1) = ({}^m n_2)({}^{m^{n_2}} n_1), \quad \forall (m, n_2) \in M * N, (n_2, n_1) \in N * N$
- (5) $(m_2m_1)^n = (m_2^{m_1 n})(m_1^n), \quad \forall (m_2, m_1) \in M * M.$

According to [Mcz2], two groupoids M and N over the same base P with a pair of actions of them on each other satisfying conditions (1)-(5) as above are called a pair of groupoids with an interaction or a “matched pair” of groupoids. Given such data, one can construct a third groupoid on the space $N * M = \{(n, m) : s_N(n) = t_M(m)\}$ over P by defining

$$\begin{aligned} s(n, m) &= s_M(m), \quad t(n, m) = t_N(n) \\ (n_1, m_1)(n_2, m_2) &= (n_1 (m_1 n_2), (m_1^{n_2}) m_2). \end{aligned}$$

The natural maps

$$\begin{aligned} M &\longrightarrow N * M : \quad m \longmapsto (\varepsilon_N(t_M(m)), m) \\ N &\longrightarrow N * M : \quad n \longmapsto (n, \varepsilon_M(s_N(n))) \end{aligned}$$

embed M and N into $N * M$ as subgroupoids.

Theorem 8.3 *Let $\sigma : G \times P \rightarrow P$ be a Poisson action of a Poisson Lie group G on an integrable Poisson manifold P . Let N be the symplectic groupoid of P . Let $N * M$ be the Lie groupoid over P as described above. Then the Lie algebroid of $N * M$ is the Lie algebroid $A = (P \times \mathfrak{g}) \bowtie T^*P$ as given in Theorem 4.1.*

We will skip the proof of this theorem. The main part of the proof consists of showing that the linearizations of the two actions of M and N on each other at their identity sections are exactly the Lie algebroid representations of $P \times \mathfrak{g}$ and T^*P on each other which were used in the construction of the Lie algebroid $A = (P \times \mathfrak{g}) \bowtie T^*P$.

Remark 8.4 For $p \in P$, denote by $(N * M)_p$ the intersection of the s -fiber and the t -fiber in $N * M$ over p . It can be identified with the space

$$(N * M)_p = \{(n, g) : n \in N, g \in G : t_N(n) = p, s_N(n) = gp\}.$$

We know from general groupoid theory that it has a group structure:

$$(n_2, g_2)(n_1, g_1) = (n_2(g_2 n_1), g_2^{\varphi(n_1)} g_1), \quad (n_2, g_2), (n_1, g_1) \in (N * M)_p.$$

The Lie algebra of $(N * M)_p$ is the Lie algebra \mathfrak{l}_p introduced in Section 4.

Remark 8.5 We remark that the space $N * M$ is an example of what is called a vacant double groupoid in [Mcz2]. On the space $N * M$, there is also a groupoid structure over M and a groupoid structure over N , namely the transformation groupoids of the actions of M and N on each other. These two groupoid structures are compatible, making $N * M$ into a double groupoid. The groupoid structure on P we just described is called the diagonal groupoid of the double groupoid $M * N$. See [Mcz2] for more details. For a detailed treatment of “matched pairs” of Lie algebroids, see [Mk].

References

- [Am] Aminou, R., Bigèbres de Lie et groupes de Lie-Poisson, Thèse Université de Lille, 1988.
- [B-B] Beilinson, A. and Bernstein, J., A proof of Jantzen conjectures, *I. M. Gelfand Seminar, Advances in Soviet Mathematics*, S. Gelfand and S. Gindikin eds. **16**, part I.
- [C-D-W] Coste, A., Dazord, P., Weinstein, A., Groupoides symplectiques (notes d’un cours de A. Weinstein), *Publ. Dept. Math., Université Claude Bernard Lyon I* (1987).
- [Da-So] Dazord, P., Sondaz, D., Groupes de Poisson affines, *Proceedings of the Seminaire Sud-Rhodanien de Geometrie*, 1989, Springer-MSRI series.
- [Dr1] Drinfel’d, V. G., Hamiltonian structures on Lie groups, Lie bialgebras and the geometric meaning of the classical Yang - Baxter equations, *Soviet Math. Dokl.* **27** (1) (1983), 68 - 71.
- [Dr2] Drinfeld, V. G., On Poisson homogeneous spaces of Poisson-Lie groups, *Theo. Math. Phys.* **95** (2) (1993), 226 - 227.
- [H-M] Higgins, P. J. and Mackenzie, K., Algebraic constructions in the category of Lie algebroids, *J. Algebra* **129** (1990), 194 - 230.
- [KS] Kosmann-Schwarzbach, Y., Poisson-Drinfel’d groups, *Topics in soliton theory and exactly solvable nonlinear equations*, M. Ablowitz, B. Fuchsteiner and M. Kruskal, eds., World Scientific, Singapore (1987), 191 - 215.
- [Kt1] Kostant, B., Lie algebra cohomology and the generalized Borel-Weil theorem, *Ann. of Math.*, **74** (2) (1961), 329 - 387.

- [Kt2] Kostant, B., Lie algebra cohomology and the generalized Schubert cells, *Ann. of Math.*, **77** (1963), 72 - 144.
- [Ku] Koszul, J. L., Crochet de Schouten-Nijenhuis et cohomologie, *Astérisque, hors série, Soc. Math. France*, Paris (1985), 257 - 271.
- [Li] Lichnerowicz, A., Les variétés de Poisson et leurs algèbres de Lie associées, *J. Diff. Geom.* **12** (1977), 253-300.
- [Lu1] Lu, J. H., Multiplicative and affine Poisson structures on Lie groups, PhD thesis, University of California, Berkeley, (1990).
- [Lu-We] Lu, J. H., Weinstein, A., Poisson Lie groups, dressing transformations, and Bruhat decompositions, *Journal of Differential Geometry* **31** (1990), 501 - 526.
- [Lu2] Lu, J. H., Momentum mappings and reductions of Poisson Lie group actions, *Proceedings of the Seminaire Sud-Rhodanien de Geometrie à Berkeley, 1989*, 1991 Springer-MSRI series.
- [Mcz1] Mackenzie, K., *Lie groupoids and Lie algebroids in differential geometry*, LMS, Lecture Note Series, **124**, Cambridge University Press, (1987).
- [Mcz2] Mackenzie, K., Double Lie algebroids and second-order geometry I, *Adv. Math.* **94** (1992), 180 - 239.
- [Mj] Majid, S., Matched pairs of Lie groups associated to solutions of the Yang-Baxter equations, *Pacific Journal of Mathematics*, **141** (2) (1990), 311 - 332.
- [Mk] Mokri, T., University of Sheffield thesis, 1995.
- [We] Weinstein, A., The local structure of Poisson manifolds, *J. Diff. Geometry* **18** (1983), 523 - 557.
- [X-M] Mackenzie, K. and Xu, P., Lie bialgebroids and Poisson groupoids, **73**, No. 2 (1994), 415 - 452.
- [STS] Semenov-Tian-Shansky, M. A., Dressing transformations and Poisson Lie group actions, *Publ. RIMS, Kyoto University* **21** (1985), 1237 - 1260.
- [Xu] Xu, P., On Poisson groupoids, *Inter. J. Math.* (1994), in press.

- [Za] Zakrzewski, S., Poisson homogeneous spaces, Proceedings of the Karpacz winter school on theoretical physics 1994, to appear. Also hep-th/9412101